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Exotic dihedral actions on homology 3-spheres

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Abstract

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Exotic actions of dihedral groups on S^{2k+1} are constructed using 3-dimensional techniques.

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This paper describes families of regular dihedral branched covers of S^3, branched over knots, which are homology spheres. These covers are notable in that components of the branch set in the cover have linking numbers other than ± 1. Such examples can be easily used to provide simple new examples of exotic (nonlinear) actions of the dihedral group on S^{2k+1}, $k > 1$.

1. Introduction

Let D_{2n} denote the dihedral group with $2n$ elements. Milnor [3] proved that D_{2n} cannot act freely on a homology sphere. The simplest class of actions of dihedral groups on spheres are hence those for which the maximal cyclic subgroup, Z_n, acts freely. Basic examples of such actions are the linear actions, obtained from linear representations of D_{2n}. (Any such representation is a direct sum of standard 2-dimensional representations.)

For these linear actions, the fixed point sets of the involutions are disjoint k-spheres in S^{2k+1}. Any two of these spheres turn out to have linking number ± 1 in S^{2n+1}. (Orienting one component of the fixed point set determines an orientation on [illegible]

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of the components, via the action of Z_n.) Recently, Davis and tom Dieck [2] have produced the first examples of nonlinear dihedral actions on S^{2k+1} for which Z_n acts freely. Their examples are distinguished from linear actions by the fact that the linking numbers of the fixed-point sets are shown to be other than ± 1.

The present paper grew out of an effort to understand the implications of [2] in dimension 3. The methods used by Davis and tom Dieck work in this dimension, only the constructed actions are on homology 3-spheres rather than on S^3. (According to Thurston [7], if a dihedral group acts on S^3, the action is necessarily linear.) Furthermore, in dimension 3, the quotient space of the homology sphere is again a homology sphere, and the quotient map is a branched covering. From a low-dimensional perspective an immediate question arises as to whether or not these quotients are S^3. So far it has not been possible to determine whether or not this is the case.

In this paper we will describe regular dihedral branched covers of S^3 by homology spheres. The branch sets in S^3 will be knots. The branch set in the cover will consist of disjoint knots with linking numbers other than ± 1. In terms of group actions, these provide examples of actions of the dihedral group on 3-dimensional homology spheres such that the fixed-point sets of the involutions are disjoint knots with exotic linking numbers.

The examples can be used to construct new examples of exotic actions on S^{2k+1} for $k > 1$: Given an action on a homology sphere, H^3, there is an action on the join, $H^3 * S^1$, formed by taking the join with a standard linear action on S^1. By the Double Suspension Theorem this join is homeomorphic to S^5. It is easily checked that the linking numbers are unchanged by the joining operation. The procedure can be repeated to produce examples in higher dimensions.

It should be noted that while our examples give new results in dimension 3, and perhaps a new perspective on the higher dimensional results, they are weaker in the sense that they hold in the topological category, whereas the Davis and tom Dieck examples yield smooth actions on homotopy spheres.

My efforts at understanding the 3-dimensional aspects of [2] were undertaken jointly with Jim Davis. Thanks are due for his assistance there, and for his help in developing this work.

2. Construction

Fix the order of the dihedral group, $2n$, with n odd.

Consider the Pretzel knot $K(p, q, r)$ illustrated in Fig. 1. The integers p, q, and r, denote half twists. The particular knot $K(2, -3, 5)$ is illustrated also. It will be shown that for appropriate choices of p, q, and r, there is a regular D_{2n} cover of S^3, branched over $K(p, q, r)$, which is a homology sphere. Appropriate choices of p, q, and r, will in fact yield an infinite family of such, distinguished by the linking numbers of the branch curves.

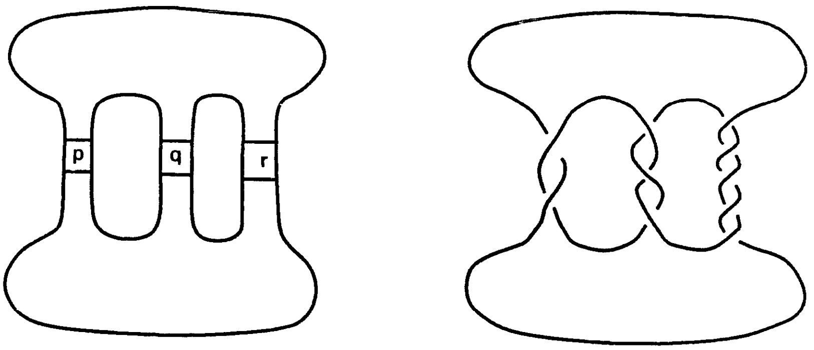

Fig. 1.

The proof that the desired covers exist, and are homology spheres, is an easy application of the methods of Akbulut and Kirby [1] for finding framed link descriptions of cyclic covers of knots. The basic methods of the Calculus of Framed Links are also needed. The reader is referred to Rolfsen [6] for background material.

The calculation of the linking numbers follows a different route. When a knot has a dihedral branched cover, there is also an associated irregular n-fold cover. The branch set in the irregular cover consists of a single curve, b_0, with branching index 1, and $(n-1)/2$ curves of index 2, which we denote by $\{b_i\}$. Perko [4] has shown that the linking numbers of the curves in the regular cover are determined by the linking numbers, $\mathrm{lk}(b_0, b_i)$. More precisely, Perko shows that the linking numbers that occur in the regular cover are exactly the set of integers $\mathrm{lk}(b_0, b_i)/2$, $0 < i \leq (n-1)/2$. Methods that are fully illustrated in Rolfsen [6, Chapter 10F] in the case of the trefoil knot are easily applied to compute the linking numbers for the knots of interest in the present work.

Arranging the cover to be a homology sphere

Consider the Pretzel knot $K_{x,y} = K(1+2xn, -2-2xn, y)$, where x is an arbitrary integer, and y is to be determined. According to [1] the 2-fold cyclic cover of S^3 branched over $K_{x,y}$ has the framed link description illustrated in Fig. 2. The $-2-2xn$ denotes the number of full twists in that illustration. As both components of the link are unknotted, the component with framing -1 can be blown down to show that the 2-fold branched cover is surgery on a knot. It is a straightforward exercise

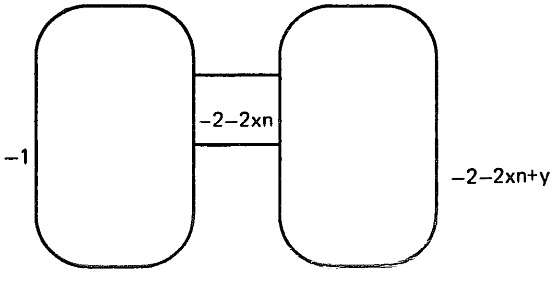

Fig. 2.

in the Framed Link Calculus to show that the cover is in fact described as surgery on the torus knot $T(2+2xn, 1+2xn)$ in S^3. The surgery coefficient is $-(4x^2n^2+6xn+2+y)$.

If $y=-n-(4x^2n^2+6xn+2)$, the 2-fold cover can be described as n-surgery on the torus knot $T(2+2xn, 1+2xn)$. Since this space has first homology Z_n, it has an n-fold regular cover. That cover is obtained from the n-fold cyclic branched cover of S^3 branched over the torus knot $T(2+2xn, 1+2xn)$ by performing surgery on the branch set. The n-fold cyclic branched cover is the Brieskorn space $\Sigma(n, 2+2xn, 1+2xn)$. As these three integers are relatively prime, the space is in fact a Brieskorn homology sphere, and the branch set is hence null homologous. That the branch set is null homologous can be seen directly by noting that any Seifert Surface for the torus knot lifts to a surface in the cover. Using a Seifert Surface, one can show that the surgery coefficient in the covering space is $+1$, and the resulting space is again a homology sphere. The composition of a 2-fold branched cover and an n-fold regular cyclic cover yields a regular dihedral branched cover.

Calculating the linking number

Once again, the argument that follows is simply a generalization of the procedure described in Rolfsen [6] for calculating the linking number of the branch curves in the 3-fold irregular cover of S^3, branched over the trefoil. We now summarize the main points of that discussion.

Consider a copy of B^3, containing two unknotted arcs, as illustrated in Fig. 3 (left). A representation of the fundamental group of the complement of the two arcs to D_{2n} determines an n-fold irregular covering space. If the representation is surjective, and the meridian of each arc maps to an element of order two, then the cover is just B^3. The two arcs lift to $(n+1)$ arcs, two with branching index 1, and the rest with branching index 2. This is illustrated in Fig. 3 (right), in the case $n=3$, The thicker arcs indicate the index 1 branch curves.

Now suppose that the two arcs are replaced with two arcs with $2n$ half twists, as illustrated in Fig. 4 (left). The old representation of the fundamental group of the boundary 2-sphere minus 4 points extends to a representation of the complement of the twisted arcs. The cover is B^3 again, and the cover on the boundary S^2 is the

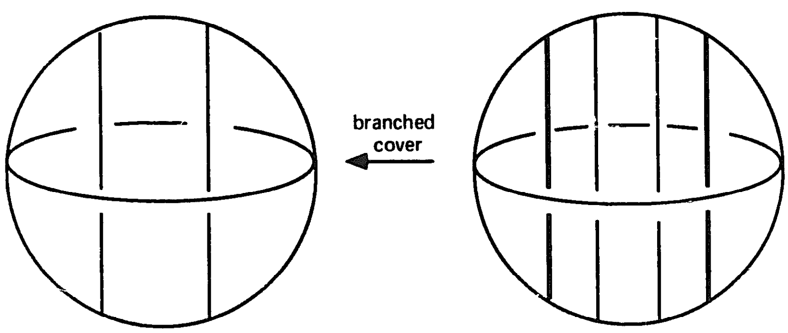

Fig. 3.

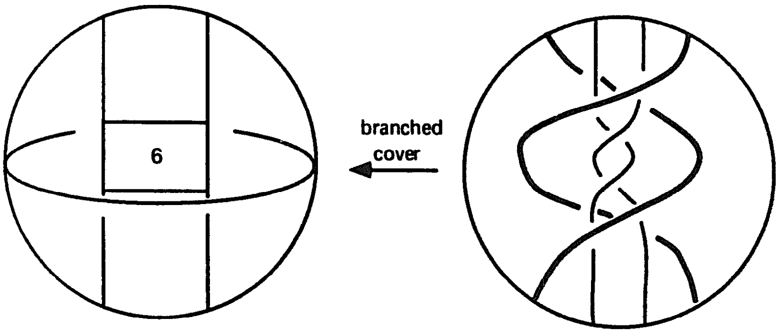

Fig. 4.

same as before. However, the branch set upstairs has received one full twist, as is illustrated in Fig. 4 (right) also.

To apply these observations to the calculation of linking numbers, suppose that there is an embedding of B^3 into S^3 which intersects our Pretzel knot in two unknotted arcs. Given that the restriction on the representation is satisfied (a fact we will shortly deal with) the above discussion shows that the knot can be changed by putting $2n$ half twists into the pair of arcs, without changing the irregular branched covering space. However, the branch set in that cover has changed; a full twist has been added to a collection of strands—Fig. 4 (right) provides the local picture.

Now the new link in the branched cover is clearly *homotopic* to the original link, but during the homotopy various components must cross. Counting the number of crossings determines the change in the linking numbers. The two arcs of index 1 lie on one component of the branch set. The rest of the arcs are paired by the component of the branch set they lie on. If the two arcs are oriented in opposite directions, then the homotopy introduces no changes in linking number. However, if the arcs were oriented in the same direction, the homotopy yields a change of 4 in the linking numbers.

For the knot of interest to us, $K(1+2xn, -2-2xn, y)$, with $y = -n-(4x^2n^2+6xn+2)$, the center strands are oriented in opposite directions, and the right and left strands are oriented in the same direction. (Note that y is odd.)

Adding $-x(2n)$ half twists to the left band, $x(2n)$ half twists to the center band, and $(2x^2n+3x)(2n)$ half twists to the right band yields the Pretzel knot $K(1, -2, -n-2)$. The linking number of any two components has been changed by $4((-x)+(2x^2n+3x)) = 8x(xn+1)$.

On the other hand, the Pretzel knot $K(1, -2, -n-2)$ is the same as the 2-stranded torus knot, $T(2, -n)$. For this knot the n-fold irregular cover is in fact S^3 and the linking numbers have been calculated. They turn out to all be -2, if n is positive, and $+2$ if n is negative. (The calculation was first done by Reidemeister [5] and used to show that these torus knots are not amphiceiral.)

There is one fact that we failed to check earlier; that is, that for each band in the Pretzel knot, the associated pair of meridians must represent to a pair of elements in the dihedral group which generate the entire group. However, if this is true for the final knot, $K(1, -2, -n-2)$, it will be true as well for the entire families of Pretzel knots.

What remains is an exercise in describing knot group representations. We state the result and leave the details to the reader. If the left and right meridians at the top of the right band of the pretzel knot represent to T and gT, then the right and left meridian at the top of the central band represent to $g^{-1}T$ and T, while the left and right meridian at the top of the left band represent to gT and $g^{-1}T$. The first two pairs clearly generate D_{2n}. The last pair does also, as n is odd.

We summarize with the following theorem. Recall that the linking numbers in the regular $2n$-fold cover are obtained from those in the n-fold irregular cover by dividing by 2. Let $n > 1$ be an odd integer.

Theorem. *For every integer x, there is a $2n$-fold regular dihedral covering of S^3 branched over the Pretzel Knot $K(p, q, r)$ with $p = 1 + 2xn$, $q = -2 - 2xn$, and $r = -n - (4x^2n^2 + 6xn + 2)$. That cover is a homology sphere, and the linking number of any two components of the branch set is $-4x(xn + 1) - 1$.*

Remarks

The construction used above can be generalized to find a wider variety of examples. For instance, the basic requirement on p and q is that $p + q = \pm 1$. More general choices of p and q yield examples in which the basic knot is a 2-bridge knot which is not necessarily a torus knot. These can be used to produce examples for which the linking numbers are not all equal.

Further high-dimensional examples of exotic dihedral actions on homotopy spheres have been announced by tom Dieck. These new examples provide realization results for linking numbers. The implications of his new methods on the 3-dimensional case are yet to be investigated.

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